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Exam I , Math 531, Spring 2014

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QUESTION 1. (i) (computational) Let $R = Z_7 \times Z_5$. Find all prime ideals of R.

Solution, Sketch : We know from HW each prime ideal of R is of the form $P \times Z_5$ for some prime P of Z_7 or of the form $Z_7 \times Q$ where Q is a prime ideal of Z_5 Since Z_7, Z_5 are fields, the prime ideals of R are $\{0\} \times Z_5$ and $Z_7 \times \{0\}$

(ii) **(computational) Let $R = Z \times Z$ and $I = 18Z \times 25Z$. Then I is an ideal of R. Find $\sqrt{I}$.**

Solution, Sketch: Let $(x, y) \in \sqrt{I}$. Then $18 \mid x^n, 25 \mid y^n$. Since 2, 3 are the prime factors of 18 and 5 is the prime factor of 25, we conclude that $6 \mid x$ and $5 \mid y$. Hence $x \in 6Z$ and $y \in 5Z$ Thus $\sqrt{I} \subseteq 6Z \times 5Z$. Let $(a, b) \in 6z \times 5Z$. Then it is clear that $(a, b)^3 = (a^3, b^3) \in 12Z \times 5Z$. Done.

(iii) Let f be a ring-epimorphism from R onto S. Given that R is a commutative ring without identity and S is a commutative ring. Can we conclude that S has no identity? The answer is no. Give an example.

Solution, Sketch: Let $R = 3Z \times Z$ is a ring with no identity and let $S = Z$ is a ring with identity, $f(3a, b) = b$ is a ring-epimorphism from R ONTO S.

(iv) Given that R is a commutative ring with $1 \neq 0$ such that $char(R) = n$.

a. Prove that there is a ring-homomorphism f from Z into R such that $Ker(f) = nZ$.
Solution, Sketch:
$f : Z \to R$ such that $f(a) = a.1_R$. Clearly f is a ring-homomorphism and $Ker(f) = nZ$. Hence Z/nZ is ring-isomorphic to Image (f).

b. If $n = 25$, prove that $|U(R)| \geq 20$ and $|Nil(R)| \geq 5$.
Solution, Sketch:
Clearly $|U(image(f)| = |U(Z_{25})| = \phi(25) = 20$. Hence $|U(R)| \geq 20$. Also note $Nil(Z_{25}) = \{0, 5, 10, 15, 20\}$. Hence $|Nil(image(f))| = 5$, and thus $|Nil(R)| \geq 5$.

c. If $n = 11$, prove that R has a subring that is a field. **Solution: Since Z_{11} is a field, $Image(f)$ is a field. Done**

(v) Let R be a commutative ring with identity and with exactly two (distinct) maximal ideals L, F such that $LF = \{0\}$.

a. Prove that each element in R is either a zero-divisor or a unit.
Solution:
By Chinese remainder theorem R is ring-isomorphic to $R/L \times R/F = F_1 \times F_2$, where F_1, F_2 are fields. By staring at the product of the two fields. the claim follows.

b. Can you tell me how many idempotents does R have? **solution: by staring at $D = F_1 \times F_2$, idempotents of D are $(0, 0), (1, 0), (0, 1), and (1, 1)$. Hence R has exactly 4 idempotents.**

c. Prove that $L = eR$ for some idempotent e of R.
Solution, Sketch:
Since L, F are co-prime, there is an $i \in L$ and an $f \in F$ such that $i + f = 1$. Since $if = 0$ by hypothesis, $i(i + f) = i \cdot 1 = i$. Hence $i^2 = i$ is an idempotent. Clearly $iR \subseteq L$. Let $x \in L$. We show $x \in iR$. Since $i + f = 1$, we have $x(i + f) = xi + xf = x \cdot 1$. Since $xf = 0$, $xi = x$. Thus $x \in iR$. Hence $L = iR$.

(vi) Let P be a prime ideal of a commutative ring R. Prove that $Nil(R) \subseteq P$

Solution, sketch:

Let $w \in Nil(R)$. Then $w^n = 0 \in P$. Let $D = R/P$. Since D is an integral domain, we have $Nil(D) = \{0\} = \{P\}$. Since $(w + P)^n = w^n + P = 0 + P = P$, we conclude that $w + P \in Nil(D)$. Thus $w + P = P$ and hence $w \in P$.

(vii) Let R be a commutative ring with 1. Let $A = R(+)R$. For $(a, b), (c, d) \in A$, define $(a, b)+(c, d) = (a+c, b+d)$ and $(a, b).(c, d) = (ac, bc + ad)$. Then we know that A is a commutative ring with 1 (Do not show that).

a. Show that $(0, r) \in Nil(A)$ for every $r \in R$. **Solution, Sketch: $(0, r)^2 = (0, 0)$. Done.**

b. Let $Q = L(+)M$ be a prime ideal of A, where L, M are some ideals of R. Show that L is a prime ideal of R and $M = R$.

Solution, Sketch: Let $xy \in L$**. Since** $(x,0)(y,0) \in Q$ **and** Q **is prime,** $(x,0) \in Q$ **or** $(y,0) \in Q$**. Thus** $x \in L$ **or** $y \in L$**.**

Let $r \in R$**. Since** $(0,r) \in Nil(A)$**, we conclude that** $(0,r) \in Q$ **by (VI). Thus** $r \in M$ **and hence** $M = R$

(viii) Let I, J be co-prime proper ideals of a commutative ring R with $1 \neq 0$. Prove that I, J^2 are co-prime ideals of R and I^2, J^2 are co-prime ideals of R.

Solution, Sketch: there is an $i \in I$ **and** $j \in J$ **such that** $i + j = 1$**. Thus** $(i + j)^2 = 1$**. Hence** $i^2 + 2ij + j^2 = 1$**. Thus** $i(i + 2j) + j^2 = 1$**. Since** $i(i + 2j) \in I$ **and** $j^2 \in J^2$ **, we are done.**

Also, $1 = (i + j)^3 = i^3 + 3i^2 j + 3ij^2 + j^3 = i^2(i + 3i) + j^2(3i + j)$**. Since** $i^2(i + 3j) \in I^2$ **and** $j^2(j + 3i) \in J^2$**, we are done**

NOTE that if you only prove that I^2 **and** J^2 **are co-prime, then surely** I **and** J^2 **are co-prime since** $I^2 \subseteq I$

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